CONTINUITY AND DIFFERENTIABILITY

BASIC CONCEPTS

1. **Continuity and Discontinuity of a Function at a Point:** A function $f(x)$ is said to be continuous at a point a of its domain if

$$\lim_{x \to a^-} f(x),\ \lim_{x \to a^+} f(x),\ f(a) \text{ exist and } \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$$

A function $f(x)$ is said to be discontinuous at $x = a$ if it is not continuous at $x = a$.

2. **Properties of Continuous Function:**
   - Every constant function is continuous function.
   - Every polynomial function is continuous function.
   - Identity function is continuous function.
   - Every logarithmic and exponential function is a continuous function.

List of Useful Formulae:

3. (*i*) $\frac{d}{dx}(x^n) = nx^{n-1}$ (*ii*) $\frac{d}{dx}(ax+b)^n = n(ax+b)^{n-1}.a$

(*iii*) $\frac{d}{dx}(e^x) = e^x$ (*iv*) $\frac{d}{dx}e^{ax} = a.e^{ax}$

(*v*) $\frac{d}{dx}a^x = a^x.\log_e a$ (*vi*) $\frac{d}{dx}a^{bx} = ba^{bx}\log_e a$

(*vii*) $\frac{d}{dx}\log_e x = \frac{1}{x}$ and $\frac{d}{dx}\log_e ax = \frac{a}{x}$

(*viii*) $\frac{d}{dx}\log_a x = \frac{1}{x.\log_e a}$ and $\frac{d}{dx}\log_a bx = \frac{b}{x.\log_e a}$

4. (*i*) $\frac{d}{dx}\sin x = \cos x$ and $\frac{d}{dx}\sin ax = a\cos ax$

(*ii*) $\frac{d}{dx}\cos x = -\sin x$ and $\frac{d}{dx}\cos ax = -a\sin ax$

(*iii*) $\frac{d}{dx}\tan x = \sec^2 x$ and $\frac{d}{dx}\tan ax = a\sec^2 ax$

(*iv*) $\frac{d}{dx}\cot x = -\text{cosec}^2 x$ and $\frac{d}{dx}\cot ax = -a\,\text{cosec}^2 ax$

(*v*) $\frac{d}{dx}\sec x = \sec x\tan x$ and $\frac{d}{dx}\sec ax = a\sec ax.\tan ax$

(*vi*) $\frac{d}{dx}\text{cosec}\, x = -\text{cosec}\, x\cot x$ and $\frac{d}{dx}\text{cosec}\, ax = -a\,\text{cosec}\, ax.\cot ax$

5. (*i*) $\frac{d}{dx}\sin^{-1}x - \frac{1}{\sqrt{1-x^2}}$ and $\frac{d}{dx}\sin^{-1}ax - \frac{a}{\sqrt{1-a^2x^2}}$

(*ii*) $\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$ and $\frac{d}{dx}\cos^{-1}ax = \frac{-a}{\sqrt{1-a^2x^2}}$

(*iii*) $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$ and $\frac{d}{dx}\tan^{-1}ax = \frac{a}{1+a^2x^2}$

(*iv*) $\frac{d}{dx}\cot^{-1}x = \frac{-1}{1+x^2}$ and $\frac{d}{dx}\cot^{-1}ax = \frac{-a}{1+a^2x^2}$

(*v*) $\frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{x^2-1}}$ and $\frac{d}{dx}\sec^{-1}ax = \frac{1}{x\sqrt{a^2x^2-1}}$

(*vi*) $\frac{d}{dx}\text{cosec}^{-1}x = \frac{-1}{x\sqrt{x^2-1}}$ and $\frac{d}{dx}\text{cosec}^{-1}ax = \frac{-1}{x\sqrt{a^2x^2-1}}$

5. **Chain Rule:** Chain rule is applied when the given function is the function of function *i.e.*, if y is a function of x, then $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$ or $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dx}$

6. **Parametric Form:** Sometimes we come across the function when both x and y are expressed in terms of another variable say t *i.e.*, $x = \phi(t)$ and $y = \psi(t)$. This form of a function is called parametric form and t is called the parameter.

7. **Rolle's Theorem:** *If $f(x)$ be a real valued function, defined in a closed interval $[a, b]$ such that:*
   (*i*) it is continuous in closed interval $[a, b]$.
   (*ii*) it is differentiable in open interval (a, b).
   (*iii*) $f(a) = f(b)$. Then there exists at least one value $c \in (a, b)$ such that $f'(c) = 0$.

8. **Lagrange's Mean Value Theorem:**
   If $f(x)$ is a real valued function defined in the closed interval $[a, b]$ such that:
   (*i*) it is continuous in the closed interval $[a, b]$.
   (*ii*) it is differentiable in the open interval (a, b).
   Then there exists at least one real value $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{(b-a)}$.

9. **Some Standard Results:**
   (*i*) (*a*) $\lim_{x \to a}\frac{x^n - a^n}{x - a} = na^{n-1}, a > 0, n \in Q$
   (*b*) $\lim_{x \to a}\frac{x^m - a^m}{x^n - a^n} = \frac{m}{n}a^{m-n}, m, n \in Q$
   (*c*) $\lim_{x \to 0}\frac{\sin x}{x} = 1$ (*d*) $\lim_{x \to 0}\cos x = 1$
   (*e*) $\lim_{x \to 0}\frac{\tan x}{x} = 1$
   (*ii*) Evaluation of limits of inverse trigonometric functions:
   (*a*) $\lim_{x \to 0}\frac{\sin^{-1}x}{x} = 1$ (*b*) $\lim_{x \to 0}\frac{\tan^{-1}x}{x} = 1$
   (*iii*) Evaluation of limits of exponential and logarithmic functions:
   (*a*) $\lim_{x \to 0}e^x = 1$ (*b*) $\lim_{x \to 0}\frac{e^x - 1}{x} = 1$
   (*c*) $\lim_{x \to 0}\frac{\log|1+x|}{x} = 1$ (*d*) $\lim_{x \to 0}\frac{(1+x)^n - 1}{x} = n$
   (*e*) $\lim_{x \to 0}\frac{a^x - 1}{x} = \log_e a$

(*iv*) Limits at infinity: This method is applied when $x \to \infty$.

Procedure to solve the infinite limits:

(*a*) Write the given expression in the form of rational function.

(*b*) Divide the numerator and denominator by highest power of x.

(*c*) Use the result $\lim_{x \to \infty} \frac{1}{x^n} = 0$, where $n > 0$.

(*d*) Simplify and get the required result.

MULTIPLE CHOICE QUESTIONS

Choose and write the correct option in the following questions.

1. The function $f : R \to R$ given by $f(x) = -|x - 1|$ is [*CBSE 2020 (65/2/1)*]

(*a*) continuous as well as differentiable at $x = 1$

(*b*) not continuous but differentiable at $x = 1$

(*c*) continuous but not differentiable at $x = 1$

(*d*) neither continuous nor differentiable at $x = 1$

2. The function $f(x) = e^{|x|}$ is [*NCERT Exemplar*]

(*a*) continuous everywhere but not differentiable at $x = 0$

(*b*) continuous and differentiable everywhere

(*c*) not continuous at $x = 0$

(*d*) none of these

3. The function $f(x) = [x]$, where $[x]$ denotes the greatest integer function, is continuous at

(*a*) 4 (*b*) –2 (*c*) 1 (*d*) 1.5

4. The number of points at which the function $f(x) = \frac{1}{x - [x]}$ is not continuous is [*NCERT Exemplar*]

(*a*) 1 (*b*) 2 (*c*) 3 (*d*) none of these

5. The function $f(x) = \begin{cases} \frac{\sin x}{x} + \cos x, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$ is continuous at $x = 0$, then the value of k is [*NCERT Exemplar*]

(*a*) 3 (*b*) 2 (*c*) 1 (*d*) 1.5

6. The value of k which makes the function defined by $f(x) = \begin{cases} \sin\frac{1}{x}, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$, continuous at $x = 0$ is

(*a*) 8 (*b*) 1 (*c*) –1 (*d*) none of these

7. The function $f(x) = \cot x$ is discontinuous on the set [*NCERT Exemplar*]

(*a*) $\{x = n\pi : n \in Z\}$ (*b*) $\{x = 2n\pi : n \in Z\}$

(*c*) $\left\{x = (2n+1)\frac{\pi}{2}; n \in Z\right\}$ (*d*) $\left\{x = \frac{n\pi}{2}; n \in Z\right\}$

8. Let $f(x) = |\sin x|$. Then [*NCERT Exemplar*]

(*a*) f is everywhere differentiable

(*b*) f is everywhere continuos but not differentiable at $x = n\pi$, $n \in Z$

(*c*) f is everywhere continuous but not differentiable at $x = (2n+1)\frac{\pi}{2}$, $n \in Z$

(*d*) none of these

9. **The function $f(x) = \dfrac{x-1}{x(x^2-1)}$ is discontinuous at** *[CBSE 2020 (65/2/2)]*

(*a*) exactly one point (*b*) exactly two points

(*c*) exactly three points (*d*) no point

10. **If $f(x) = x^2 \sin\dfrac{1}{x}$, where $x \neq 0$, then the value of the function f at $x = 0$, so that the function is continuous at $x = 0$, is** *[NCERT Exemplar]*

(*a*) 0 (*b*) –1 (*c*) 1 (*d*) None of these

11. **The function $f(x) = |x| + |x-1|$ is**

(*a*) continuous at $x = 0$ as well as at $x = 1$. (*b*) continuous at $x = 1$ but not at $x = 0$.

(*c*) discontinuous at $x = 0$ as well as at $x = 1$. (*d*) continuous at $x = 0$ but not at $x = 1$.

12. **The function $f(x) = \dfrac{4-x^2}{4x-x^3}$ is**

(*a*) discontinuous at only one point (*b*) discontinuous at exactly two points

(*c*) discontinuous at exactly three points (*d*) none of these

13. **The set of points where the functions f given by $f(x) = |x-3| \cos x$ is differentiable is**

(*a*) R (*b*) $R - \{3\}$ (*c*) $(0, \infty)$ (*d*) none of these

14. **Differential coefficient of $\sec(\tan^{-1}x)$ w.r.t. x is** *[NCERT Exemplar]*

(*a*) $\dfrac{x}{\sqrt{1+x^2}}$ (*b*) $\dfrac{x}{1+x^2}$ (*c*) $x\sqrt{1+x^2}$ (*d*) $\dfrac{1}{\sqrt{1+x^2}}$

15. **If $u = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$ and $v = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$, then $\dfrac{du}{dv}$ is**

(*a*) $\dfrac{1}{2}$ (*b*) x (*c*) $\dfrac{1-x^2}{1+x^2}\{4, -4\}, \phi$ (*d*) 1

16. **If $y = \log\sqrt{\tan x}$, then the value of $\dfrac{dy}{dx}$ at $x = \dfrac{\pi}{4}$ is**

(*a*) 0 (*b*) 1 (*c*) $\dfrac{1}{2}$ (*d*) ∞

17. **If $y = \sqrt{\sin x + y}$, then $\dfrac{dy}{dx}$ is equal to**

(*a*) $\dfrac{\cos x}{2y-1}$ (*b*) $\dfrac{\cos x}{1-2y}$ (*c*) $\dfrac{\sin x}{1-2y}$ (*d*) $\dfrac{\sin x}{2y-1}$

18. **The function $f(x) = |x| + |x-2|$ is**

(*a*) differentiable at $x = 0$ and at $x = 2$ (*b*) differentiable at $x = 0$ but not at $x = 2$.

(*c*) not differentiable at $x = 0$ and at $x = 2$. (*d*) none of these

19. **The function given by $f(x) = \tan x$ is discontinuous on the set**

(*a*) $\{x : x = 2n\pi, n \in Z\}$ (*b*) $\{x : x = (n-1)\pi, n \in Z\}$

(*c*) $\{x : x = n\pi, n \in Z\}$ (*d*) $\{x : x = (2n+1)\dfrac{\pi}{2}, n \in Z\}$

20. **The derivative of $\tan x$ w.r.t. $\sin x$ is**

(*a*) $\tan^2 x$ (*b*) $\sec x$ (c) $\dfrac{\tan x}{\sin x}$ (*d*) $\sec^3 x$

21. **The function $f(x) = \dfrac{x^2-x-6}{x-3}$ is not defined for $x = 3$. In order to make $f(x)$ continuous at $x = 3$, $f(3)$ should be defined as**

(*a*) 1 (*b*) 3 (*c*) 5 (*d*) none of these

22. **If $f(x) = x - 3$ and $g(x) = \frac{x^2}{3} + 1$, then which of the following can be a discontinuous function?**

(*a*) $f(x) + g(x)$ (*b*) $f(x) \cdot g(x)$ (*c*) $f(x) - g(x)$ (*d*) $\frac{g(x)}{f(x)}$

23. **The set of points where the function f given by $|3x - 2| \sin x$ is differentiable is**

(*a*) R (*b*) $(0, \infty)$ (*c*) $R - \left\{\frac{2}{3}\right\}$ (*d*) none of these

24. **Let $f(x) = \begin{cases} \cos [x], & x \geq 0 \\ |x| + a, & x < 0 \end{cases}$ where $[x]$ denotes the greatest integer $\leq x$. If $\lim_{x \to 0} f(x)$ exists then a is equal to**

(*a*) 1 (*b*) 3 (*c*) 0 (*d*) none of these

25. **Let $f(x) = \begin{cases} x[x-1], & 0 \leq x < 2 \\ [x](x-1), & 2 \leq x < 3 \end{cases}$ where $[x]$ denotes the greatest integer $\leq x$ then**

(*a*) $f(x)$ is continuous everywhere.

(*b*) $f(x)$ is not continuous at $x = 1$ and $x = 2$

(*c*) $f(x)$ is not differentiable at infinite points

(*d*) none of these

26. **$f : [-2a, 2a] \to R$ is an odd function such that the left hand derivative at $x = a$ is zero and $f(x) = f(2a - x) \;\forall\; x \in (a, 2a)$. Then its left hand derivative at $x = -a$ is**

(*a*) 0 (*b*) a (*c*) 1 (*d*) does not exist.

27. **The function $f(x) = |x - 3|, x \in R$ is**

(*a*) is continuous and differentiable everywhere.

(*b*) is not continuous but differentiable at $x = 3$

(*c*) is continuous but not differentiable at $x = 3$

(*d*) none of these

28. **If $f(x) = x^n$, then the value of**

$$f(1) - \frac{f'(1)}{1!} + \frac{f''(1)}{2!} - \frac{f'''(1)}{3!} \ldots\ldots\ldots\ldots + \frac{(-1)^n f^n(1)}{n!} \text{ is}$$

(*a*) 1 (*b*) 0 (*c*) 2^n (*d*) 2

29. **If $x = e^{y + e^{y + \ldots \text{ to } \infty}}$, $x > 0$, then $\frac{dy}{dx}$ is**

(*a*) $\frac{1}{x}$ (*b*) $\frac{x}{1+x}$ (*c*) $\frac{1-x}{x}$ (*d*) none of these

30. **Let $f(x) = \frac{1 - \tan x}{4x - \pi}$, $x \neq \frac{\pi}{4}$, $x \in \left[0, \frac{\pi}{2}\right]$. If $f(x)$ is continuous in $\left[0, \frac{\pi}{2}\right]$, then $f\left(\frac{\pi}{4}\right)$ is**

(a) $\frac{1}{2}$ (*b*) $-\frac{1}{2}$ (*c*) 1 (*d*) -1

31. **The value of p and q for which the function**

$$f(x) = \begin{cases} \dfrac{\sin (p+1)x + \sin x}{x} & , x < 0 \\ q & , x = 0 \\ \dfrac{\sqrt{x + x^2} - \sqrt{x}}{x^{3/2}} & , x > 0 \end{cases}$$ **is continuous for all $x \in R$, are**

(*a*) $p = \frac{1}{2}, q = \frac{3}{2}$ (*b*) $p = \frac{5}{2}, q = \frac{7}{2}$ (*c*) $p = -\frac{3}{2}, q = \frac{1}{2}$ (*d*) none of these

32. The function $f(x) = \begin{cases} \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ **at** $x = 0$

(*a*) is continuous. (*b*) has removable discontinuity.

(*c*) has jump discontinuity. (*d*) has oscillating discontinuity.

33. The function $f(x) = \begin{cases} 2 - x, & \text{if } x < 2 \\ 2 + x, & \text{if } x \geq 2 \end{cases}$ **at** $x = 2$

(*a*) is continuous. (*b*) has removable discontinuity.

(*c*) has jump discontinuity. (*d*) has oscillating discontinuity.

34. The function $f(x) = \begin{cases} \frac{\sin x}{x} + \cos x, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$ **at** $x = 0$

(*a*) is continuous. (*b*) has removable discontinuity.

(*c*) has jump discontinuity. (*d*) has oscillating discontinuity.

35. If $f(x) = |x| + |x - 2|$, **then**

(*a*) $f(x)$ is continuous at $x = 0$ but not at $x = 2$.

(*b*) $f(x)$ is continuous at $x = 0$ and at $x = 2$.

(*c*) $f(x)$ is continuous at $x = 2$ but not at $x = 0$.

(*d*) None of these.

36. The function $f(x) = \frac{1}{x - 1}$ **at** $x = 1$.

(*a*) is continuous (*b*) has removable discontinuity.

(*c*) has jump discontinuity. (*d*) has asymptotic discontinuity.

37. The vertical asymptotes to curve $y = \frac{e^x}{x}$ **is**

(*a*) $x = 1$ (*b*) $x = 0$

(*c*) $x = 2$ (*d*) Curve has no any asymptotes.

38. An oblique asymptote to the curve $y = x + e^{-x} \sin x$ **is**

(*a*) $y = x + e$ (*b*) $y = x$ (*c*) $y = x + \frac{1}{\pi}$ (*d*) none of these

39. The function $f(x) = \begin{cases} x^m \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ **at** $x = 0$ **is continuous if**

(*a*) $m \geq 0$ (*b*) $m > 0$ (*c*) $m < 0$ (*d*) none of these

40. The function $f(x) = \begin{cases} \frac{1 - \cos 10x}{x^2}, & \text{if } x < 0 \\ m, & \text{if } x = 0 \\ \frac{\sqrt{x}}{\sqrt{625 + \sqrt{x}} - 25}, & \text{if } x > 0 \end{cases}$ **is continuous at** $x = 0$**, if the value of** m **is**

(*a*) 25 (*b*) 50 (*c*) –25 (*d*) none of these

41. The set of all points where $f(x) = \sec 2x + \operatorname{cosec} 2x$ **is discontinuous is**

(*a*) $\{n\pi : n = 0, \pm 1, \pm 2, \ldots\ldots\ldots\}$ (*b*) $\left\{\frac{n\pi}{2} : n = 0, \pm 1, \pm 2, \ldots\ldots\ldots\right\}$

(*c*) $\left\{\frac{(2n+1)\pi}{4} ; n - 0, \pm 1, \pm 2, \ldots\ldots\ldots\right\}$ (*d*) $\left\{\frac{n\pi}{4} : n = 0, \pm 1, \pm 2, \ldots\ldots\ldots\right\}$

42. **If $f(x) = x^n$, then the value of**

$$f(1)+\frac{f(1)}{1!}+\frac{f^2(1)}{2!}+\frac{f^3(1)}{3!}+ \ldots\ldots\ldots\ldots + \frac{f^n(1)}{n!},\text{ where } f^r(1)$$

is the r^{th} derivative of $f(x)$ w.r.t. x.

(*a*) 1 (*b*) n (*c*) 2^n (*d*) none of these

43. **If $f(x), g(x), h(x)$ are polynomials in x of degree 2 and**

$$F(x)=\begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x)\end{vmatrix},\text{ then } F'(x) \text{ is equal to}$$

(*a*) –1 (*b*) 2 (*c*) 0 (*d*) none of these

44. **The derivative of $f(\tan x)$ w.r.t. $g(\sec x)$ at $x = \frac{\pi}{4}$, where $f'(1) = g'(\sqrt{2}) = 4$ is**

(*a*) $\sqrt{2}$ (*b*) $\frac{1}{\sqrt{2}}$ (*c*) 1 (*d*) none of these

45. **If g is inverse function of f and $f'(x) = \sin x$, then $g'(x)$ is**

(*a*) $\sin(g(x))$ (*b*) $\sin^{-1} x$ (*c*) $\frac{1}{\sqrt{1-x^2}}$ (*d*) $\text{cosec}\,(g(x))$

46. **If $y = \left(\frac{x}{n}\right)^{nx}\left(1+\log\frac{x}{n}\right)$, $y'(n)$ is given by**

(*a*) $\frac{n^2+1}{n}$ (*b*) $\frac{1}{n}$ (*c*) $\left(\frac{1}{n}\right)^n$ (*d*) $\left(\frac{1}{n}\right)^n\left(\frac{n^2+1}{n}\right)$

47. **If $f(x) = \log\sqrt{\tan x}$, then the value of $f'(x)$ at $x = \frac{\pi}{4}$ is**

(*a*) ∞ (*b*) 1 (*c*) 0 (*d*) $\frac{1}{2}$

48. **Let y be a function of x such that $\log(x + y) - 2xy = 0$, then $y'(0)$ is**

(*a*) 0 (*b*) 1 (*c*) $\frac{1}{2}$ (*d*) $\frac{3}{2}$

49. **If $x \cos y + y \cos x = \pi$ then $y''(0)$ is**

(*a*) π (*b*) $-\pi$ (*c*) 0 (*d*) 1

50. **Let $f(x)=\begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3\end{vmatrix}$, where p is constant, then find $f''(0)$.**

(*a*) p (*b*) 0 (*c*) $p + p^3$ (*d*) $p + p^2$

51. **If $y = f\left(\frac{3x+4}{5x+6}\right)$ and $f'(x) = \tan x^2$ then $\frac{dy}{dx}$ is equal to**

(*a*) $-2\tan\left(\frac{3x+4}{5x+6}\right)^2 \times \frac{1}{(5x+6)^2}$ (*b*) $\tan x^2$

(*c*) $f\left(\frac{3\tan x^2+4}{5\tan x^2+6}\right)$ (*d*) none of these

52. **The set of all points where the function $f(x) = x|x|$ is differentiable is**

(*a*) $(-\infty, \infty)$ (*b*) $(-\infty, 0) \cup (0, \infty)$ (*c*) $(0, \infty)$ (*d*) $[0, \infty)$

53. **Let $f(x) = \cos^{-1}(\cos x)$ then $f(x)$ is**

(*a*) continuous at $x = \pi$ and not differentiable at $x = \pi$.

(*b*) continuous at $x = -\pi$

(*c*) differentiable at $x = 0$

(*d*) differentiable at $x = \pi$

54. Let $f(x) = x - |x|$ then $f(x)$ is

(*a*) differentiable $\forall\, x \in \mathbb{R}$

(*b*) continuous $\forall\, x \in \mathbb{R}$ and not differentiable at $x = 0$

(*c*) neither continuous nor differentiable at $x = 0$

(*d*) discontinuous at $x = 0$

55. Let $f(x) = \begin{cases} -1, & -2 \le x < 0 \\ x^2 - 1, & 0 < x \le 2 \end{cases}$ **and** $g(x) = |f(x)| + f(|x|)$

then the number of points at which $g(x)$ is non-differentiable is

(*a*) at most one point (*b*) two (*c*) exactly one point (*d*) infinite

56. Let $f(x)$ be differentiable function such that

$$f\left(\frac{x+y}{1-xy}\right) = f(x) + f(y)\ \forall\ x \text{ and } y.$$

If $\lim_{x \to 0} \frac{f(x)}{x} = \frac{1}{3}$ **then $f(1)$ is equal to**

(*a*) $\frac{\pi}{4}$ (b) $\frac{\pi}{12}$ (*c*) $\frac{\pi}{6}$ (*d*) None of these

57. Let $f(x) = \begin{cases} 1 - 3x, & x < 0 \\ 3, & x = 0 \\ x^2 + 3, & x > 0 \end{cases}$ **then at $x = 0$**

(*a*) $f(x)$ is continuous from left

(*b*) $f(x)$ is continuous

(*c*) $f(x)$ is right continuous

(*d*) $f(x)$ has removable discontinuity

58. Let $f(x) = |\sin x|;\ 0 \le x \le 2\pi$ then

(*a*) $f(x)$ is differentiable function at infinite number of points

(*b*) $f(x)$ is non-differentiable at 3 points and continuous everywhere.

(*c*) $f(x)$ is discontinuous everywhere

(*d*) $f(x)$ is discontinuous at 3 points

59. Let $f(x) = \begin{cases} \left[\tan\left(\frac{\pi}{4} + x\right)\right]^{\frac{1}{x}}, & x \ne 0 \\ k, & x = 0 \end{cases}$

then the value of k such that $f(x)$ holds continuity at $x = 0$ is

(*a*) e (*b*) $\frac{1}{e^2}$ (*c*) e^2 (*d*) None of these

60. Let $f(x) = x^2|x|$ then the set of values, where $f(x)$ is three times differentiable, is

(*a*) Infinite (*b*) 2 (*c*) 3 (*d*) None of these

61. Let $f(x) = \begin{cases} -3 & -3 \le x < 0 \\ x^2 - 3 & 0 \le x \le 3 \end{cases}$ **and $g(x) = |f(x)| + f(|x|)$ then which of the following is true?**

(*a*) At $x = 0$, $g(x)$ is continuous as well as differentiable and at $x = \sqrt{3}$, $g(x)$ is continuous but not differentiable.

(*b*) At $x = \sqrt{3}$, $g(x)$ is neither continuous nor differentiable

(*c*) At $x = 2$, $g(x)$ is neither continuous nor differentiable

(*d*) None of these

62. **Let $f(x) = x^{3/2} - \sqrt{x^3 + x^2}$ then**

(*a*) LHD at $x = 0$ exists but RHD at $x = 0$ does not exists
(*b*) $f(x)$ is differentiable at $x = 0$
(*c*) RHD at $x = 0$ exists but LHD at $x = 0$ does not exists
(*d*) None of these

63. **Number of points at which $f(x) = \dfrac{1}{\log|x|}$ is discontinuous is**

(*a*) 2 (*b*) 3 (*c*) 1 (*d*) 4

64. **If $f(x) = \dfrac{\sin 4\pi[\pi^2 x]}{7 + [x]^2}$, [•] denotes the greatest integer function, then $f(x)$ is**

(*a*) continuous for all x but $f'(x)$ does not exist
(*b*) discontinuous at some x
(*c*) $f''(x)$ exists for all x
(*d*) $f'(x)$ exists but $f''(x)$ does not exist for some value of x.

65. **The function $f(x) = \begin{cases} \dfrac{\sin^3 x^2}{x}, & x \neq 0 \\ 0 & x = 0 \end{cases}$ is**

(*a*) continuous but not derivable at $x = 0$
(*b*) neither continuous nor differentiable at $x = 0$
(*c*) continuous and differentiable at $x = 0$
(*d*) none of these

66. **Let $f(x) = \sin^{-1}\left(\dfrac{1 + x^2}{2x}\right)$ then $f(x)$ is**

(*a*) differentiable at $x = 1$
(*b*) continuous $\forall\, x \in \mathbb{R}$
(*c*) neither continuous nor differentiable at $x = 1$
(*d*) continuous but not differentiable at $x = 1$

67. **Let $f(x) = \begin{cases} x, & x < 1 \\ 2 - x, & 1 \le x \le 2 \\ -2 + 3x - x^2, & x > 2 \end{cases}$ then $f(x)$ is**

(*a*) differentiable at $x = 1$ (*b*) differentiable at $x = 2$
(*c*) differentiable at $x = 1$ and $x = 2$ (*d*) none of these

68. **Let $f(x) = |\log|x||$ then**

(*a*) $f(x)$ is continuous in its domain but not differentiable at $x = \pm 1$
(*b*) $f(x)$ is continuous and differentiable for all x in its domain
(*c*) $f(x)$ is neither continuous nor differentiable at $x = \pm 1$
(*d*) all of these

69. **Let $g(x) = \begin{cases} e^{2x}, & x < 0 \\ e^{-2x}, & x \ge 0 \end{cases}$ then $g(x)$ does not satisfy the condition**

(*a*) continuous $\forall\, x \in \mathbb{R}$
(*b*) not differentiable at $x = 0$
(*c*) continuous $\forall\, x \in \mathbb{R}$ and non differentiable at $x = \pm 1$
(*d*) none of these

70. Let $f(x) = [x]^2 + \sqrt{\{x\}}$, where $[\cdot]$ and $\{\cdot\}$ respectively denotes the greatest integer and fractional part functions, then

(*a*) $f(x)$ is continuous at all integral points

(*b*) $f(x)$ is non differentiable $\forall\ x \in \mathbb{Z}$

(*c*) $f(x)$ is discontinuous $\forall x \in \mathbb{Z} - \{1\}$

(*d*) $f(x)$ is continuous and differentiable at $x = 0$

71. Find the value of a if the function $f(x)$ defined by

$$f(x) = \begin{cases} 2x-1, & x<2 \\ a, & x=2 \\ x+1, & x>2 \end{cases}$$ **is continuous at $x = 2$.**

(*a*) 3 (*b*) – 3 (*c*) 0 (*d*) 4

72. If the function $f(x)$ defined by

$$f(x) = \begin{cases} \dfrac{\log(1+ax) - \log(1-bx)}{x}, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$$ **is continuous at $x = 0$, find k.**

(*a*) a (*b*) b (*c*) $a + b$ (*d*) 0

73. If function $f(x) = e^{-|x|}$ is

(*a*) continuous everywhere but not differentiable at $x = 0$

(*b*) continuous and differentiable everywhere

(*c*) not continuous at $x = 0$

(*d*) None of these.

74. If $x^y = e^{x-y}$ find $\dfrac{dy}{dx}$.

(*a*) $\dfrac{\log x}{(1+\log x)^2}$ (*b*) $\dfrac{x}{\log x}$ (*c*) $\dfrac{\log x}{(1-\log x)^2}$ (*d*) None of these

75. If $x^y = y^x$, find $\dfrac{dy}{dx}$.

(*a*) $x \log x$ (*b*) $\dfrac{y}{x}\cdot\left(\dfrac{x\log y - y}{y\log x - x}\right)$ (*c*) 0 (*d*) None of these

76. If $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \ldots \infty}}}$, find $\dfrac{dy}{dx}$.

(*a*) $\dfrac{\cos x}{2y+1}$ (*b*) $\dfrac{\cos x}{2y-1}$ (*c*) 0 (*c*) None of these

77. If $x = \sqrt{a^{\sin^{-1}t}}$, $y = \sqrt{a^{\cos^{-1}t}}$, $a > 0$ and $-1 < t < 1$, then $\dfrac{dy}{dx}$ is :

(*a*) $\dfrac{y}{x}$ (*b*) $\dfrac{x}{y}$ (*c*) $\dfrac{-y}{x}$ (*d*) None of these

78. Derivative of $\tan^{-1}\left(\dfrac{1+2x}{1-2x}\right)$ w.r.t $\sqrt{1+4x^2}$:

(*a*) $\dfrac{1}{2x\sqrt{1+4x^2}}$ (*b*) $\dfrac{1}{x\sqrt{1+x^2}}$ (*c*) $\dfrac{1}{4x\sqrt{1+2x^2}}$ (*d*) $\dfrac{1}{2x\sqrt{1-4x^2}}$

79. If $f(x) = \begin{vmatrix} x+a^2 & ab & ac \\ ab & x+b^2 & bc \\ ac & bc & x+c^2 \end{vmatrix}$, find $f'(x)$.

(*a*) $3x^2 - 2x(a^2 + b^2 + c^2)$ (*b*) $3x^2 + 2x(a^2 + b^2 + c^2)$

(c) 0 (d) None of these

80. If $y = x^x$ find $\frac{d^2y}{dx^2}$.

(a) $x^x\left\{(1+\log x)^2 - \frac{1}{x}\right\}$ (b) $x^x\left\{(1+\log x)^2 + \frac{1}{x}\right\}$

(c) 0 (d) $x^x\left\{(1-\log x)^2 + \frac{1}{x}\right\}$

81. Find $\frac{d^2y}{dx^2}$, if $x = at^2$, $y = 2at$.

(a) $\frac{-1}{2at^3}$ (b) $\frac{1}{2at^2}$ (c) $\frac{-1}{2at^2}$ (d) 0

82. Determine the value of the constant k so that the function

(a) $\frac{3}{2}$ (b) $\frac{3}{4}$ (c) $\frac{1}{2}$ (d) 0

Answers

1. (c)	**2.** (a)	**3.** (d)	**4.** (d)	**5.** (b)	**6.** (d)
7. (a)	**8.** (b)	**9.** (c)	**10.** (a)	**11.** (a)	**12.** (c)
13. (b)	**14.** (a)	**15.** (d)	**16.** (b)	**17.** (a)	**18.** (c)
19. (d)	**20.** (d)	**21.** (c)	**22.** (d)	**23.** (c)	**24.** (a)
25. (b)	**26.** (a)	**27.** (c)	**28.** (b)	**29.** (c)	**30.** (b)
31. (c)	**32.** (d)	**33.** (c)	**34.** (b)	**35.** (b)	**36.** (d)
37. (b)	**38.** (b)	**39.** (b)	**40.** (b)	**41.** (d)	**42.** (c)
43. (c)	**44.** (a)	**45.** (d)	**46.** (a)	**47.** (b)	**48.** (b)
49. (a)	**50.** (b)	**51.** (a)	**52.** (a)	**53.** (a)	**54.** (b)
55. (c)	**56.** (b)	**57.** (c)	**58.** (b)	**59.** (c)	**60.** (a)
61. (a)	**62.** (c)	**63.** (b)	**64.** (c)	**65.** (c)	**66.** (c)
67. (b)	**68.** (a)	**69.** (c)	**70.** (c)	**71.** (a)	**72.** (c)
73. (a)	**74.** (a)	**75.** (b)	**76.** (b)	**77.** (c)	**78.** (a)
79. (b)	**80.** (b)	**81.** (a)	**82.** (b)		

CASE-BASED QUESTIONS

Choose the correct option in the following questions.

1. Read the following and answer any four questions from (i) to (v).

A potter made a mud vessel, where the shape of the pot is based on $f(x) = |x-3| + |x-2|$, where $f(x)$ represents the height of the pot.

[*CBSE Question Bank*]

Answer the questions given below.

(i) When $x > 4$ What will be the height in terms of x ?

(a) $x - 2$ (b) $x - 3$ (c) $2x - 5$ (d) $5 - 2x$

(ii) **Will the slope vary with x value?**

(a) Yes (b) No

(c) may or may not vary (d) none of these

(iii) **What is $\frac{dy}{dx}$ at $x = 3$**

(a) 2 (b) −2

(c) Function is not differentiable (d) 1

(iv) **When the x value lies between (2, 3) then the function is**

(a) $2x - 5$ (b) $5 - 2x$ (c) 1 (d) 5

(v) **If the potter is trying to make a pot using the function $f(x) = [x]$, will he get a pot or not? Why?**

(a) Yes, because it is a continuous function

(b) Yes, because it is not continuous

(c) No , because it is a continuous function

(d) No , because it is not continuous

Sol. (i) We have, $f(x) = |x-3| + |x-2|$

When $x > 4$

$\therefore \quad f(x) = (x-3) + (x-2) = 2x - 5$

$\therefore$ Option (c) is correct.

(ii) Yes, because when $2 < x < 3$, we have

$$f(x) = -(x-3) + (x-2) = 1$$

$\Rightarrow \quad$ Slope $= f'(x) = 0$

but when $x > 3$, we have

$$f(x) = x - 3 + x - 2 = 2x - 5$$

then slope $= f'(x) = 2$

$\therefore$ Option (a) is correct.

(iii) At $x = 3$

$$\text{L.H.D} = \lim_{\lambda \to 0} \frac{f(3-\lambda) - f(3)}{-\lambda} = \lim_{\lambda \to 0} \frac{-(3-\lambda-3) + (3-\lambda-2) - 1}{-\lambda}$$

$$= \lim_{\lambda \to 0} \frac{\lambda + 1 - \lambda - 1}{-\lambda} = \lim_{\lambda \to 0} \frac{0}{-\lambda} = 0$$

$$\text{R.H.D} = \lim_{\lambda \to 0} \frac{f(3+\lambda) - f(3)}{\lambda}$$

$$= \lim_{\lambda \to 0} \frac{(3+\lambda-3) + (3+\lambda-2) - 1}{\lambda}$$

$$= \lim_{\lambda \to 0} \frac{\lambda + 1 + \lambda - 1}{\lambda} = 2$$

L.H.D $\neq$ R.H.D at $x = 3$

$\therefore \quad f(x)$ is not differentiable at $x = 3$

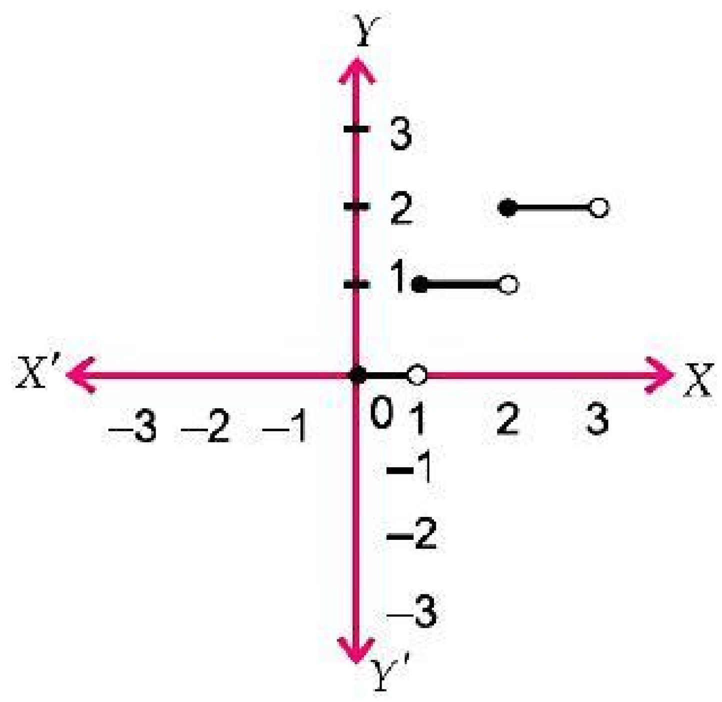
$\therefore$ Option (c) is correct.

(iv) When $2 < x < 3$, we have

$$f(x) = -(x-3) + (x-2) = 1$$

∴ Option (c) is correct.

(v) We have the function $f(x) = [x] \leq x$, where x is an integer.



It is not a continuous function, so the potter can not make a pot using the function $f(x) = [x]$.

∴ Option (d) is correct.

ASSERTION-REASON QUESTIONS

In the following questions, a statement of Assertion (A) is followed by a statement of Reason (R). Choose the correct answer out of the following choices.

(a) Both A and R are true and R is the correct explanation of A.

(b) Both A and R are true but R is not the correct explanation of A.

(c) A is true but R is false.

(d) A is false and R is also false.

1. **Assertion (A) :** If $f(x).g(x)$ is continuous at $x = a$. then $f(x)$ and $g(x)$ are separately continuous at $x = a$.

   Reason (R) : Any function $f(x)$ is said to be continuous at $x = a$, if $\lim_{h \to 0} f(a+h) = f(a)$.

2. **Assertion (A) :** If $f(x)$ and $g(x)$ are two continuous functions such that $f(0) = 3$, $g(0) = 2$, then $\lim_{x \to 0} \{f(x) + g(x)\} = 5$.

   Reason (R) : If $f(x)$ and $g(x)$ are two continuous functions at $x = a$ then $\lim_{x \to a} \{f(x) + g(x)\} = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$.

3. **Assertion (A) :** $|\sin x|$ is a continuous function.

   Reason (R) : If $f(x)$ and $g(x)$ both are continuous functions, then $gof(x)$ is also a continuous function.

4. **Assertion (A) :** If $y = \sin x$, then $\frac{d^3y}{dx^3} = -1$ at $x = 0$.

   Reason (R) : If $y = f(x) \,.\, g(x)$, then $\frac{dy}{dx} = f(x) \,.\, \frac{d}{dx} g(x) + g(x) \frac{d}{dx} f(x)$.

Answers

1. (d) **2.** (a) **3.** (a) **4.** (b)

HINTS/SOLUTIONS OF SELECTED MCQS

1. We have,

$$f(x) = -|x-1| = \begin{cases} x-1, & \text{if } x \le 1 \\ -(x-1), & \text{if } x > 1 \end{cases}$$

At $x = 1$

LHL $= \lim_{h\to 0} f(1-h) = \lim_{h\to 0} \{(1-h)-1\} = 0$

RHL $= \lim_{h\to 0} f(1+h) = \lim_{h\to 0} -(1+h-1) = 0$

$f(1) = 1 - 1 = 0$

$\therefore$ LHL = RHL = $f(0)$ $\Rightarrow$ $f(x)$ is continuous every where.

Now, at $x = 1$

$$\text{LHD} = \frac{d}{dx}(x-1) = 1\,; \quad \text{RHD} = \frac{d}{dx}\{-(x-1)\} = -1$$

LHD $\neq$ RHD

$\therefore$ $f(x)$ is not differentiable of $x = 1$.

$\therefore$ Option (*c*) is correct.

3. $f(x) = [x]$

Let $c \in Z$

LHL $= \lim_{x\to c^-} f(x) = \lim_{h\to 0} f(c-h) = \lim_{h\to 0} [c-h]$

$= c - 1$ $[\because\ h > 0 \Rightarrow -h < 0 \Rightarrow c-h < c \text{ and } c \in Z]$

RHL $= \lim_{x\to c^+} f(x) = \lim_{h\to 0} f(c+h) = \lim_{h\to 0} [c+h]$

$= c$ $[\because\ h > 0 \Rightarrow c+h > c \text{ and } c \in Z]$

As LHL $\neq$ RHL for $c \in Z$

$\therefore$ $f(x)$ is not continuous at and $c \in Z$ and $c \in Z$ is an orbitrary so $f(x)$ is not continuous at all the integers.

$\therefore$ In the given option (*d*) is the correct option

4. $f(x) = \dfrac{1}{x-[x]}$

$\because$ $x - [x] = 0 \ \forall \ x \in Z$

$f(x)$ is not defined $\forall$ $x \in Z$ number of points of discontinuity is infinite.

$\therefore$ Option (*d*) is correct.

6. Indeed $\lim_{x\to 0} \sin\dfrac{1}{x}$ does not exist.

$\therefore$ Option (d) is correct.

7. We know that, $f(x) = \cot x$ is continuous in $R - \{n\pi : n \in Z\}$.

Since, $f(x) = \cot x = \dfrac{\cos x}{\sin x}$ [since, $\sin x = 0$ at $\{n\pi, n \in Z\}$]

Hence, $f(x) = \cot x$ is discontinuous on the set $\{x = n\pi : n \in Z\}$.

$\therefore$ Option (*a*) is correct.

8. $f(x) = |\sin x|$

As sin x is continuous everywhere so $f(x) = |\sin x|$ is continuous everywhere.

And $\sin x = 0$ for $x = n\pi$, $n \in Z$

$\therefore$ $f(x) = |\sin x|$ is not differentiable at $x = n\pi : n \in Z$

$\therefore$ Option (*b*) is correct.

10. $\because f(x) = x^2 \sin\left(\frac{1}{x}\right)$, where $x \neq 0$; $\because \lim_{x \to 0} f(x) = 0$

Hence, value of the function *f* at $x = 0$, so that it is continuous at $x = 0$ is 0.

$\therefore$ Option (*a*) is correct.

12. $f(x) = \dfrac{4 - x^2}{4x - x^3}$

$f(x)$ is discontinuous where $4x - x^3 = 0$

i.e., $x(4 - x^2) = 0$ i.e., $x(2 + x)(2 - x) = 0$ i.e., At $x = 0, -2, 2$

Hence $f(x)$ is discontinuous at exactly three points.

$\therefore$ Option (*c*) is correct.

13. $f(x) = |x - 3|\cos x = g(x)h(x)$ where $g(x) = |x - 3|$ and $h(x) = \cos x$

$h(x) = \cos x$ is differentiable everywhere

But $g(x) = |x - 3|$ is differentiable everywhere except at $x = 3$.

$\therefore$ $f(x) - g(x)\, h(x)$ is differentiable everywhere except at $x - 3$.

$\therefore$ $f(x) = g(x)\, h(x)$ is differentiable at $x \in R - \{3\}$.

$\therefore$ Option (*b*) is correct.

15. $u = \sin^{-1}\left(\frac{2x}{1 + x^2}\right) = 2\tan^{-1} x \Rightarrow \frac{du}{dx} = \frac{2}{1 + x^2}$

$v = \tan^{-1}\left(\frac{2x}{1 - x^2}\right) = 2\tan^{-1} x \Rightarrow \frac{dv}{dx} = \frac{2}{1 + x^2}$

$$\therefore \frac{du}{dv} = \frac{du/dx}{dv/dx} = \frac{2/1 + x^2}{2/1 + x^2} = 1$$

$\therefore$ Option (*d*) is correct.

16. We have, $y = \log\sqrt{\tan x}$

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{\tan x}} \times \frac{1}{2\sqrt{\tan x}} \times \sec^2 x$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sec^2 x}{2\tan x} \Rightarrow \frac{dy}{dx}_{\text{at } x = \frac{\pi}{4}} = \frac{(\sqrt{2})^2}{2 \times 1} = \frac{2}{2} = 1$$

$\therefore$ Option (*b*) is correct.

18. LHD (at $x = 0$) $\neq$ RHD (at $x = 0$)

Also, LHD (at $x = 2$) $\neq$ RHD (at $x = 2$)

Therefore, $f(x) = |x| + |x - 2|$ is not differentiable at $x = 0$ and $x = 2$.

$\therefore$ Option (c) is correct.

19. Since, the value of function $f(x) = \tan x$ is $\pm\infty$ $\forall x = (2n + 1)\frac{\pi}{2}, n \in Z$.

Hence $f(x) = \tan x$ is discontinuous on the set $\left\{x : x = (2n + 1)\frac{\pi}{2}, n \in Z\right\}$.

$\therefore$ Option (*d*) is correct.

20. Let $y = \tan x$ and $t = \sin x$

$\Rightarrow \quad \frac{dy}{dx} = \sec^2 x$ and $\frac{dt}{dx} = \cos x$

Now, derivative of $\tan x$ w.r.t. $\sin x = \frac{dy}{dt} = \frac{dy/dx}{dt/dx} = \frac{\sec^2 x}{\cos x} = \sec^3 x$

$\therefore$ Option (*d*) is correct.

22. $\because \quad \frac{g(x)}{f(x)} = \frac{\frac{x^2}{3}+1}{x-3} = \frac{x^2+3}{3(x-3)} \Rightarrow \lim_{x\to 3}\frac{g(x)}{f(x)} = \infty$

$\Rightarrow$ Function $\frac{g(x)}{f(x)}$ is discontinuous at $x = 3$.

$\therefore$ Option (*d*) is correct.

24. $\because \lim_{x\to 0} f(x)$ exists $\Rightarrow \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$

$\Rightarrow \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(0+h)$

$\Rightarrow \lim_{h\to 0} f(-h) = \lim_{h\to 0} f(h)$

$\Rightarrow \lim_{h\to 0}\{|-h|+a\} = \lim_{h\to 0} \cos[h]$

$\Rightarrow a = \cos 0 \quad \Rightarrow \quad a = 1$

$\therefore$ Option (*a*) is correct.

25. Since, $0 \le x < 1 \Rightarrow -1 \le x-1 < 0 \Rightarrow [x-1] = -1$

$1 \le x < 2 \Rightarrow 0 \le x-1 < 1 \Rightarrow [x-1] = 0$

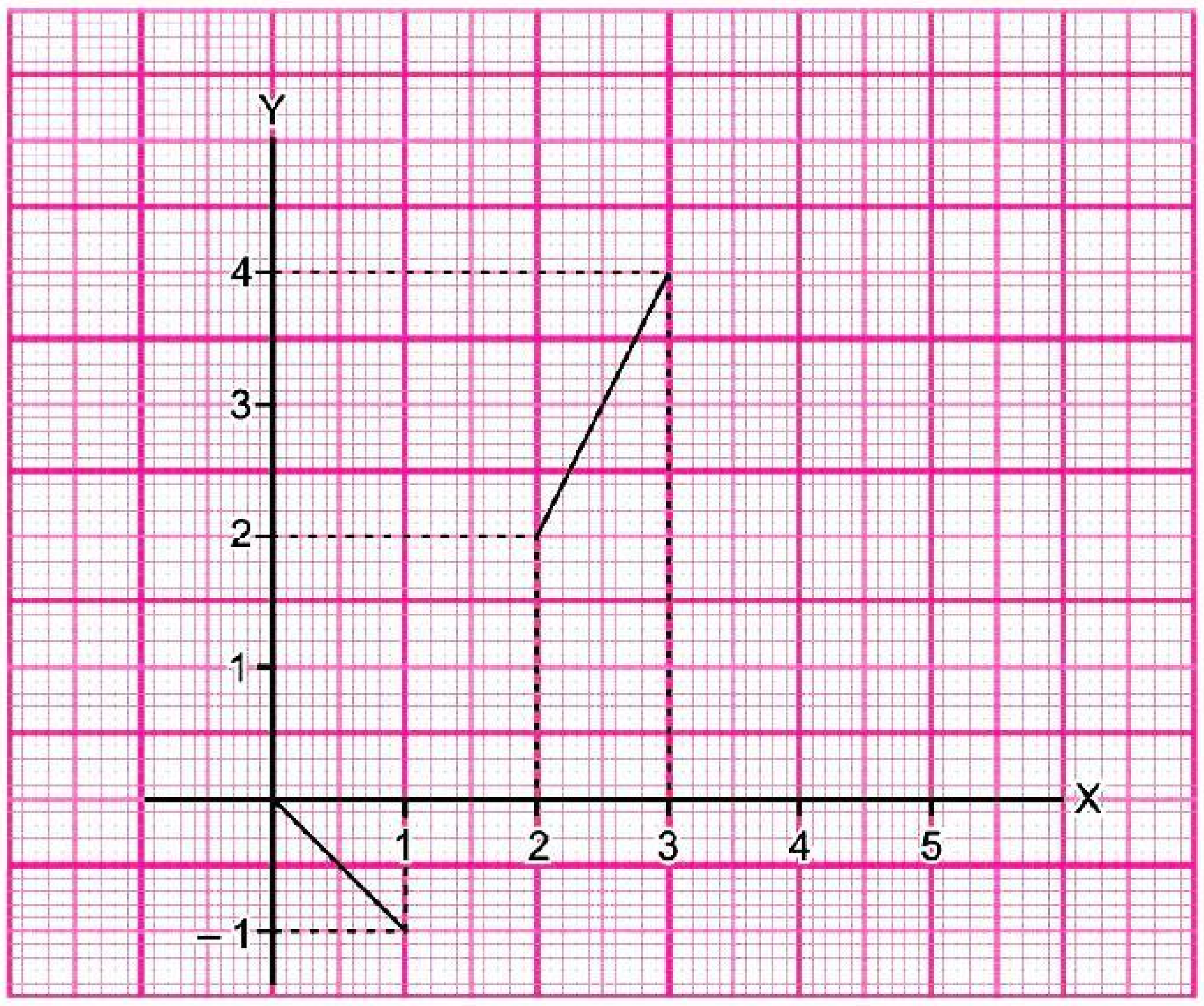
And $2 \le x < 3 \Rightarrow [x] = 2$

Therefore, given function may be written as

$$f(x) = \begin{cases} -x, & 0 \le x < 1 \\ 0, & 1 \le x < 2 \\ 2(x-1), & 2 \le x < 3 \end{cases}$$

From the graph it is obvious that

$f(x)$ is not continuous at $x = 1$ and 2, and thus not differentiable



$\therefore$ Option (*b*) is correct.

29. Given expression is

$$x = e^{y + e^{y + \ldots\ldots\ldots \text{to } \infty}} \quad \Rightarrow \quad x = e^{y+x}$$

Taking log on both sides, we have

$$\log x = \log e^{y+x} \quad \Rightarrow \quad \log x = y + x$$

Differentiating, w.r.t. x, we get

$$\frac{1}{x} = \frac{dy}{dx} + 1 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{x} - 1$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1-x}{x}$$

$\therefore$ Option (c) is correct.

32. $$f(x) = \begin{cases} \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

$\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} \sin\left(\frac{1}{x}\right)$ does not exist.

But value of $\sin\left(\frac{1}{x}\right)$ oscillates between -1 and 1.

$\therefore$ Option (d) is correct.

36 $$f(x) = \frac{1}{x-1}$$

As $\lim\limits_{x \to 1} f(x) = \lim\limits_{x \to 1} \frac{1}{x-1}$ does not exist.

$\therefore$ $f(x)$ has asymptotic discontinuity.

$\therefore$ Option (d) is correct.

37. $y = \frac{e^x}{x}$ is the curve which has vertical asymptotes at $x = 0$.

$\therefore$ Option (b) is correct.

38. $$y = x + e^{-x} \sin x = x + \frac{\sin x}{e^x}$$

$\because \quad \lim\limits_{x \to \infty} \frac{\sin x}{e^x} = 0$

As $-1 \leq \sin x \leq 1$

$\therefore \quad -e^{-x} \leq e^{-x} \sin x \leq e^{-x}$

$\lim\limits_{x \to \infty} \pm e^{-x} = 0$

$\therefore \lim\limits_{x \to \infty} e^{-x} \sin x = 0$

$\therefore$ Line $y = x$ is oblique asymptote to the given curve.

$\therefore$ Option (b) is correct.

41. $$f(x) = \sec 2x + \operatorname{cosec} 2x = \frac{1}{\cos 2x} + \frac{1}{\sin 2x}$$

$\cos 2x = 0 \Rightarrow 2x = (2n+1)\frac{\pi}{2} \Rightarrow x = (2n+1)\frac{\pi}{4}$ where $n = 0, \pm 1, \pm 2, \ldots\ldots$

And $\sin 2x = 0 \Rightarrow 2x = n\pi \Rightarrow x = \frac{n\pi}{2}$ where $n = 0, \pm 1, \pm 2, \pm 3, \ldots.$

None of the options (*a*), (*b*) and (*c*) are correct.

$\therefore$ Option (*d*) is correct.

42. $f(x) = x^n$

$$f(1) + \frac{f^1(1)}{\lfloor 1} + \frac{f^2(1)}{\lfloor 2} + \frac{f^3(1)}{\lfloor 3} + ... + \frac{f^n(1)}{\lfloor n}$$

$$= 1 + \frac{n}{\lfloor 1} + \frac{n(n-1)}{\lfloor 2} + \frac{n(n-1)(n-2)}{\lfloor 3} + ... + \frac{n(n-1)...1}{\lfloor n}$$

$= (1 + 1)^n = 2^n$ [By using Binomial expansion]

Hence option (*c*) is correct.

43. $F(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix}$

where $f(x)$, $g(x)$, $h(x)$ are polynomial of degree 2.

$\therefore \quad f'''(x) = 0 = g'''(x) = h'''(x)$

$$F'(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ f''(x) & g''(x) & h''(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f'''(x) & g'''(x) & h'''(x) \end{vmatrix}$$

$= 0 + 0 + 0 = 0$

$\therefore$ Option (*c*) is correct.

45. Given that $g(x)$ is the inverse of $f(x)$

$\therefore \quad (fog)(x) = x \Rightarrow f(g(x)) = x$

$\therefore \quad f'(g(x))g'(x) = 1$

$$\Rightarrow \quad g'(x) = \frac{1}{f'(g(x))}$$

We have $f'(x) = \sin x$ so $f'(g(x)) = \sin(g(x))$

$$\therefore \quad g'(x) = \frac{1}{\sin(g(x))} = \operatorname{cosec}(g(x))$$

$\therefore$ Option (*d*) is correct.

47. $f(x) = \log\sqrt{\tan x}$

$$f'(x) = \frac{1}{\sqrt{\tan x}} \times \frac{1}{2\sqrt{\tan x}} \times \sec^2 x \qquad = \frac{1}{2\tan x} \times \sec^2 x$$

$$f'\left(\frac{\pi}{4}\right) = \frac{1}{2 \times 1} \times (\sqrt{2})^2 = \frac{2}{2} = 1$$

Option (*b*) is correct.

50. $f(x) = \begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$

$$f'(x) = \begin{vmatrix} 3x^2 & \cos x & -\sin x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix} + \begin{vmatrix} x^3 & \sin x & \cos x \\ 0 & 0 & 0 \\ p & p^2 & p^3 \end{vmatrix} + \begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 3x^2 & \cos x & -\sin x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix} + 0 + 0$$

$$\Rightarrow \quad f'(x) = \begin{vmatrix} 3x^2 & \cos x & -\sin x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$$

$$\Rightarrow \quad f''(x) = \begin{vmatrix} 6x & -\sin x & -\cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix} + \begin{vmatrix} 3x^2 & \cos x & -\sin x \\ 0 & 0 & 0 \\ p & p^2 & p^3 \end{vmatrix} + \begin{vmatrix} 3x^2 & \cos x & -\sin x \\ 6 & -1 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 6x & -\sin x & -\cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$$

Similarly, $f'''(x) = \begin{vmatrix} 6 & -\cos x & \sin x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$

$$\Rightarrow \quad f'''(0) = \begin{vmatrix} 6 & -1 & 0 \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix} = 0$$

$\therefore$ Option (b) is correct.

52. $f(x) = x|x| = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}$

At $x = 0$

LHD = $\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{-x^2 - 0}{x}$

$= \lim_{x \to 0^-} (-x) = 0$

RHD = $\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{x^2 - 0}{x} = \lim_{x \to 0^+} (x) = 0$

LHD = RHD so f is differentiable at $x = 0$.

And $f(x)$ is also differentiable at $x \neq 0$.

$\therefore$ Set of all points where given function $f(x)$ is differentiable is $R = (-\infty, \infty)$.

Option (a) is correct.

54. $f(x) = x - |x| = g(x) + h(x)$

where $g(x) = x$, $h(x) = -|x|$

As $g(x)$ and $h(x)$ are both continuous $\forall\, x \in \mathbb{R}$

$\therefore f(x)$ is continuous $\forall\, x \in \mathbb{R}$

And $g(x)$ is differentiable $\forall\, x \in \mathbb{R}$

but $h(x)$ is not differentiable at $x = 0$

$\therefore f(x) = g(x) + h(x)$ is not differentiable at $x = 0$

$\therefore$ Option (b) is correct.

56. We have, $f\left(\frac{x+y}{1-xy}\right) = f(x) + f(y) \;\forall\; x, y \in \mathbb{R}$

Since it is of the form $\tan^{-1}\left(\frac{x+y}{1-xy}\right) = \tan^{-1}x + \tan^{-1}y$

Let $f(x) = A\tan^{-1}x$

$$\lim_{x\to 0}\frac{f(x)}{x} = \lim_{x\to 0}\frac{A\tan^{-1}x}{x} = \frac{1}{3}$$

$\Rightarrow A\lim_{x\to 0}\frac{\tan^{-1}x}{x} = \frac{1}{3} \quad \Rightarrow \quad A\times 1 = \frac{1}{3} \quad \Rightarrow \quad A = \frac{1}{3}$

$\therefore f(x) = \frac{1}{3}\tan^{-1}x$

$\Rightarrow f(1) = \frac{1}{3}\tan^{-1}(1) = \frac{1}{3}\times\frac{\pi}{4} = \frac{\pi}{12}$

$\therefore$ Option (*b*) is correct.

57. LHL $= \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(1-3x) = 1$

RHL $= \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(x^2+3) = 3$

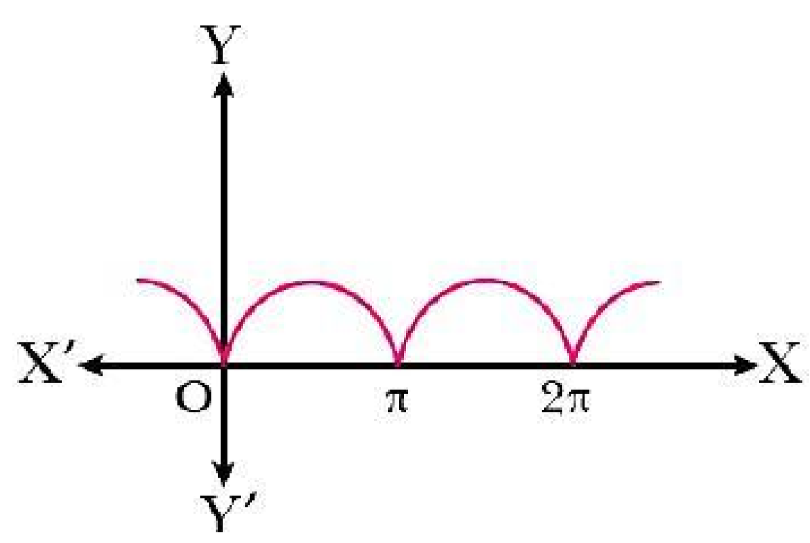
$f(0) = 3$, LHL $\neq$ RHL $= f(0)$

Here $\lim_{x\to 0^+} f(x) = f(0)$

$\Rightarrow f(x)$ is right continuous but discontinuous from left.

$\therefore$ Option (*c*) is correct.

58.



It is clear from graph that $f(x)$ is continuous everywhere in $0 \le x \le 2\pi$. And has sharp edge at $x = 0, \pi, 2\pi$ so it is not differentiable at $x = 0, \pi, 2\pi$.

Because it has no unique tangents.

$\therefore$ Option (*b*) is correct.

59. $\lim_{x\to 0} f(x) = \lim_{x\to 0}\left[\tan\left(\frac{\pi}{4}+x\right)\right]^{\frac{1}{x}} = \lim_{x\to 0}\left[\frac{1+\tan x}{1-\tan x}\right]^{\frac{1}{x}}$

$$= \lim_{x\to 0}\left[(1+\tan x)^{\frac{1}{\tan x}}\right]^{\frac{\tan x}{x}} \times \lim_{x\to 0}\left[(1-\tan x)^{-\frac{1}{\tan x}}\right]^{\frac{\tan x}{x}}$$

$$= e\times e = e^2 \quad \left[\because \lim_{x\to 0}[1+x]^{\frac{1}{x}} = e\right]$$

$\because f(x)$ is continuous at $x = 0$.

$\therefore \lim_{x\to 0} f(x) = f(0)$

$\Rightarrow e^2 = k \qquad \Rightarrow k = e^2$

$\therefore$ Option (c) is correct.

62. $f(x) = x^{3/2} - \sqrt{x^3 + x^2}$

Domain of $f = [0, \infty)$

So, LHD at $x = 0$ does not exists.

$\therefore$ Option (c) is correct.

63. $f(x) = \dfrac{1}{\log|x|}$

$f(x)$ is not defined for $x = 0, -1, 1$

$\therefore f(x)$ is not continuous at $x = 0, -1, 1$

$\therefore$ Option (b) is correct.

64. We have $f(x) = \dfrac{\sin 4\pi\,[\pi^2 x]}{7 + [x]^2}$

We know that $[\pi^2 x]$ is an integer for every x.

$\therefore 4\pi[\pi^2 x]$ is an integral multiple of π.

$\therefore \sin 4\pi[\pi^2 x] = 0$ and $7 + [x]^2 \neq 0\ \forall\ x$

$\therefore f(x) = 0\ \forall\ x$

$\Rightarrow f(x)$ is a constant function so $f'(x), f''(x), f'''(x), ...f^{n}(x)$ exists $\forall\ x$.

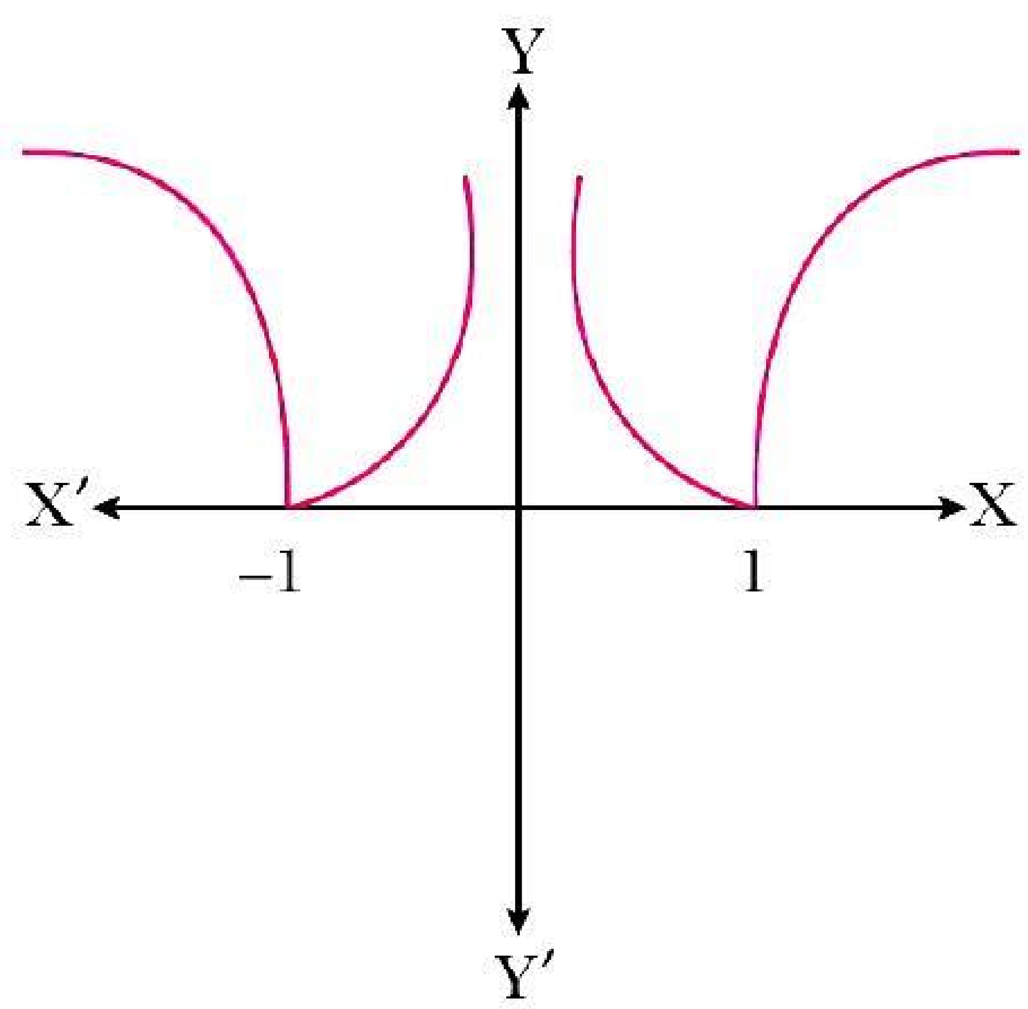
$\therefore$ Option (c) is correct.

66. We know that domain of $\sin^{-1}x$ is $|x| < 1$ and so $\sin^{-1}\left(\dfrac{1+x^2}{2x}\right)$ is defined only for $|x| < 1$.

Hence, $f(x)$ is neither continuous nor differentiable at $x = 1$

$\therefore$ Option (c) is correct.

68. $y = |\log|x||$



From the graph, $f(x)$ is continuous in its domain but LHD at $x = 1$ is negative.

RHD at $x = 1$ is positive (as shown from the graph)

as LHD at $x = 1 \neq$ RHD at $x = 1$

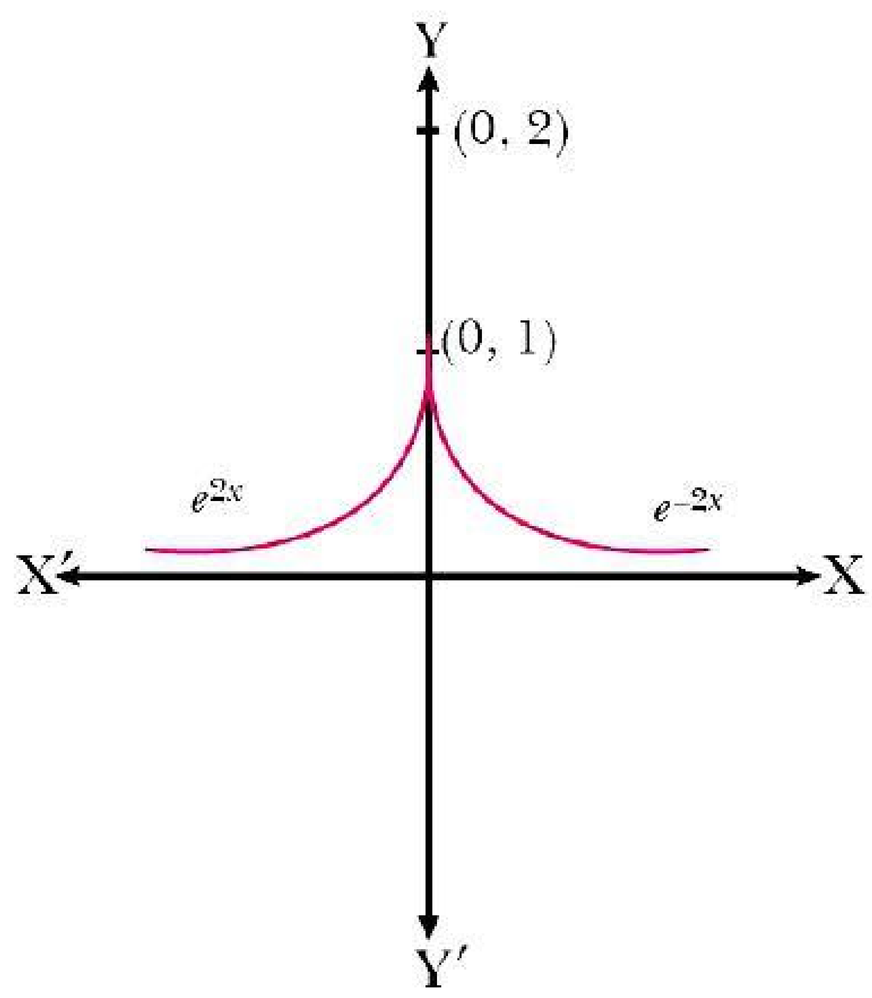
$\therefore f(x)$ is not differentiable at $x = 1$.

Also LHD at $x = -1$ is negative (as shown from the graph) and RHD at $x = -1$ is positive.

$\therefore f(x)$ is not differentiable at $x = -1$.

$\therefore$ Option (a) is correct.

69.



Given $g(x)=\begin{cases} e^{2x}, & x<0 \\ e^{-2x}, & x \geq 0 \end{cases}$

$g'(x)=\begin{cases} 2e^{2x}, & x<0 \\ -2e^{-2x}, & x \geq 0 \end{cases}$

$\therefore$ LHD at $x = 0, g'(0) = 2e^{2 \times 0} = 2e^0 = 2$

RHD at $x = 0, g'(0) = -2e^0 = -2 \times 1 = -2$

As LHD $\neq$ RHD at $x = 0$

$\therefore$ $g(x)$ is not differentiable at $x = 0$.

Again RHL = $\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} e^{-2x} = e^0 = 1$

LHL = $\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} e^{2x} = e^0 = 1$

$g(0) = e^0 = 1$

As LHL = RHL = $f(0)$

$\therefore$ $g(x)$ is continuous $\forall\ x \in \mathbb{R}$

$\therefore$ Option (*c*) is correct.

71. $f(x)=\begin{cases} 2x-1, & x<2 \\ a, & x=2 \\ x+1, & x>2 \end{cases}$

RHL = $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x+1) = 3$

LHL = $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (2x-1) = 4 - 1 = 3$

$\because$ Since f is continuous at $x = 2$ so LHL = RHL = $f(2)$

$\Rightarrow 3 = a \Rightarrow a = 3$

$\therefore$ Option (*a*) is correct.

73. $f(x) = e^{-|x|} = \dfrac{1}{e^{|x|}}$ is continuous everywhere but not differentiable at $x = 0$

$\therefore$ Option (*a*) is correct.

74. $x^y = e^{x-y}$

Taking logarithm on both sides, we get

$y \log x = (x - y) \log e = x - y$

$\Rightarrow (1 + \log x)y = x \Rightarrow y = \dfrac{x}{1 + \log x}$

$$\frac{dy}{dx}=\frac{(1+\log x)\times 1-x\left(0+\frac{1}{x}\right)}{(1+\log x)^2}=\frac{1+\log x-1}{(1+\log x)^2}=\frac{\log x}{(1+\log x)^2}$$

$\therefore$ Option (*a*) is correct.

75. $x^y = y^x \Rightarrow y\log x = x\log y$

$$\frac{y}{x}+\log x\frac{dy}{dx}=\frac{x}{y}\frac{dy}{dx}+\log y$$

$$\Rightarrow\left(\log x-\frac{x}{y}\right)\frac{dy}{dx}=\left(\log y-\frac{y}{x}\right)=\frac{x\log y-y}{x}$$

$$\Rightarrow\frac{dy}{dx}=\frac{x\log y-y}{y\log x-x}\times\frac{y}{x}=\frac{y}{x}\times\frac{x\log y-y}{y\log x-x}$$

$\therefore$ Option (*b*) is correct.

76. Let $y=\sqrt{\sin x+\sqrt{\sin x+\sqrt{\sin x+...}}}$

$\Rightarrow y=\sqrt{\sin x+y}$ should be out of square root

$\Rightarrow y^2=\sin x+y\Rightarrow y^2-y=\sin x$

Differentiate with respect to x, we get

$$\therefore 2y\frac{dy}{dx}-\frac{dy}{dx}=\cos x$$

$$\Rightarrow(2y-1)\frac{dy}{dx}=\cos x\ \Rightarrow\frac{dy}{dx}=\frac{\cos x}{\sqrt{2y-1}}$$

$\therefore$ Option (*b*) is correct.

80. $y=x^x=e^{x\log x}$

$$\therefore\ \frac{dy}{dx}=e^{x\log x}\left(x\times\frac{1}{x}+\log x\right)$$

$$=e^{x\log x}(1+\log x)$$

$$\therefore\frac{d^2y}{dx^2}=e^{x\log x}\left(\frac{1}{x}\right)+(1+\log x)e^{x\log x}(1+\log x)$$

$$=\frac{x^x}{x}+(1+\log x)^2e^x\log x$$

$$=x^{x-1}+(1+\log x)^2x^x=x^x\left\{(1+\log x)^2+\frac{1}{x}\right\}$$

$\therefore$ Option (*b*) is correct.

81. $x=at^2\Rightarrow\frac{dx}{dt}=2at$

$$y=2at\Rightarrow\frac{dy}{dt}=2a$$

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{2a}{2at}=\frac{1}{t}$$

$$\therefore\frac{d^2y}{dx^2}=-\frac{1}{t^2}\times\frac{dt}{dx}=-\frac{1}{t^2}\times\frac{1}{2at}=-\frac{1}{2at^3}$$

$\therefore$ Option (*a*) is correct.